Commodity Market Models with Stochastic Volatility and Jumps

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Introduction

This workshop will address advanced modelling and numerical techniques that are fitting for today’s market situation, where volatility and jump risk premia have become more pronounced than ever. The participants will meet best practices as well as get insight into cutting edge techniques.

What we will cover:

- Potentials and limits of affine jump diffusion models
- Option pricing using characteristic functions and Fourier inversion
- Risk management by capturing the dynamics of commodity prices
- Simulating future volatility paths and calculating VaR
- Assessing the impact of spot price jumps on commodity variance swaps
- Probing links to interest rates, credit, equity, and foreign exchange modelling
Background and Affiliations

- Doctorate in Mathematics (cont.)

- Finalyse – Risk and Valuation Consultant (2005-)
- FORTIS – Senior Model Validation Quant (2006-)
- KU Leuven – Doctoral Researcher (2007-)
1. Modelling Financial Asset Price Dynamics

2. Affine Jump-Diffusion Processes and the Fourier Inversion

3. Modelling the Dynamics of Commodity Curves

4. Pricing Volatility and Jump Derivatives

5. Hedging and Risk Management

6. Questions
Financial modelling

- What is the price for a derivative?
- What is my trading position?
- How to aggregate positions on different maturities?
- How is my day-to-day PnL moving and why?
- How will my PnL distribute assuming static hedge?
- What is my VaR and what do the stress tests tell? Consolidated figures incorporating currency and interest rate risks
- How to extrapolate futures curves (control the market making at risk management, segregated markets – long term for structured products) – liquidity issues
- Consistent aggregation of risk figures – aggregate greeks calculated by the same model
- Unspanned stochastic volatility – what are the hedging assets needed?
Black and Scholes (1973) diffusions

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- One risk factor
- Daily asset price returns are normally distributed independent random variables
- Complete market, perfect hedge exists
- But, on the market options with different strikes and maturities are quoted with different volatility parameter
Merton (1976) jump-diffusions

\[ dS_t = \mu S_t dt + \sigma S_t dW_t + \left( e^J - 1 \right) S_t dN_t \]

- Still one factor model, but incomplete market
- Skewed and heavy-tailed distributions for daily asset price returns, but they are still independent
- Due to central limit theorem, the yearly returns are close to normally distributed, no fat tail
- Implied volatility surface should not change according to the model
Heston (1993) stochastic volatility

\[ dS_t = \mu_S S_t dt + \sqrt{\nu_t} S_t dW^S_t \]
\[ d\nu_t = \kappa (\theta - \nu_t) + \sigma \sqrt{\nu_t} dW^\nu_t \]
\[ \langle dW^S_t, dW^\nu_t \rangle = \rho \]

- Already two risk factors
- Volatility factor has co-dependence
- Implied volatility smile and long-term skew
- Not able to fit short-term skew and long-term skew at the same time, instable parameters
- Short-term skew depends on the constant rho, thus no stochastic leverage effect
Dupire (1994) local volatility

\[ dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t \]

- One risk factor
- Continues asset price path
- Complete market
- Wrong volatility dynamics
- Forward skew disappear
Bates (1996) stochastic volatility + jumps

\[ dS_t = \mu S_t dt + \sqrt{\nu_t S_t} dW_t^S + \left(e^J - 1\right) S_t dN_t \]
\[ d\nu_t = \kappa \left(\theta - \nu_t\right) + \sigma \sqrt{\nu_t} dW_t^{\nu} \]

- Still two risk factors
- Can explain the extra skew in the short-term
- Jumps take off some pressure from vol-of-vol and mean-reversion
- Short-term skew depends on the constant rho and on time-homogenous jumps, thus still no stochastic leverage effect
Quoting the smile by delta

-25% put delta
ATM
+25% call delta
Forex implied volatilities

EURUSDV3M
EURUSDV6M
EURUSDV1Y
Forex risk reversals

EURUSD25R3M Currency
EURUSD25R6M Currency
EURUSD25R1Y Currency
Relative risk reversals and spot

Graph showing relative risk reversals and spot for EURUSD 25R3M / V3M, EURUSD 25R6M / V6M, and EURUSD 25R1Y / V1Y.
Stochastic leverage models

- Empirically the level and the slope of the volatility smirk fluctuate largely independently
  - Forex: distributions are usually skewed to the weaker currency, the direction of the strength, thus the sign of the skew may change
  - Equity: default expectation, risk-averseness and jump-to-default premium are stochastic, thus the level of skew may change
  - Rates: anticipated central bank actions may imply significant skew, also the sign of the skew may change
  - Commodity: upside jumps are sometime more probable than downside jumps, also the sign of the skew may change

- Focus on the stochastic correlation between asset and variance returns
Fang (2000) stochastic jump intensity

\[ dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S + (e^J - 1) S_t \, dN \]

\[ d\nu_t = \kappa_{\nu} (\theta_{\nu} - \nu_t) + \sigma_{\nu} \sqrt{\nu_t} \, dW_t^\nu \]

\[ d\lambda_t = \kappa_{\lambda} (\theta_{\lambda} - \lambda_t) + \sigma_{\lambda} \sqrt{\lambda_t} \, dW_t^\lambda \]

- Three risk factors
- Stochastic jump-intensity implies stochastic short-term skew
- Candidate to analyse and describe volatility and jump effects on the risk premia valued by the market
- Link the equity and credit markets (jump-to-default modelling)
- Not necessarily realistic for modelling equity indices
Multi-factor stochastic volatility
- Wishart processes with various correlation structures
- Permit a separate fit of both short-term and long-term skew
- Candidates to analyse and describe volatility and stochastic correlation effects on the risk premia valued by the market
Christoffersen, Heston and Jacobs (2007)

\[
dS_t = \mu S_t dt + \sqrt{\nu_{1,t}} S_t dW^1_t + \sqrt{\nu_{2,t}} S_t dW^2_t
\]

\[
d\nu_{1,t} = \left( \eta_1 - \kappa_1 \nu_{1,t} \right) dt + \sigma_1 \sqrt{\nu_{1,t}} dW^3_t
\]

\[
d\nu_{2,t} = \left( \eta_2 - \kappa_2 \nu_{2,t} \right) dt + \sigma_2 \sqrt{\nu_{2,t}} dW^4_t
\]

- Various correlation structures
- Permit a separate fit of both short-term and long-term skew
- Candidates to analyse and describe volatility and stochastic correlation effects on the risk premia valued by the market
- I could not find it applicable to model stochastic leverage
- No mean-reversion may produce persistent volatility
Schwartz and Smith (2000)

\[
\ln(S_t) = \chi_t + \xi_t
\]
\[
d\chi_t = -\kappa \chi_t \, dt + \sigma \chi \, dW^\chi_t
\]
\[
d\xi_t = \mu \xi \, dt + \sigma \xi \, dW^\xi_t
\]

- Two-factor model to incorporate mean-reversion in spot prices – short-term deviations from the equilibrium price
- It is also a stochastic convenience yield model
- Short-term returns are normally distributed
- It assumes that the implied volatility term structure, which is decreasing in maturity, is constant in time
Crosby (2006) mean-reverting jumps

\[
\frac{dF_{t,T}}{F_{t,T}} = \mu_{t,T} dt + \sum_{k=1}^{K} \sigma_k(t, T) dW_t^k + \sum_{m=1}^{M} \left( e^{J_m(t,T)} - 1 \right) dN_t^m
\]

\[
\sigma_k(t, T) = a_{k,t} + b_{k,t} \exp\left( -\int_t^T c_{k,t}(u) \, du \right)
\]

\[
J_m(t, T) = d_{m,t} \exp\left( -\int_t^T f_{m,t}(u) \, du \right)
\]

- Multi-factor jump-diffusion model with mean-reversion in commodity prices – generates stochastic convenience yields
- Consistent with any initial futures term structure
- Allows the prices of long-dated futures contracts to jump by smaller magnitudes than short-dated futures contracts
Trolle and Schwartz (2008) unspanned vol

\[
dS_t = y_{t,t} S_t dt + \sigma_{s1} \sqrt{\nu_{1,t}} S_t dW_t^1 + \sigma_{s1} \sqrt{\nu_{2,t}} S_t dW_t^2
\]

\[
dy_{t,T} = \mu_{t,T} dt + \sigma_{y1} \sqrt{\nu_{1,t}} dW_t^3 + \sigma_{y2} \sqrt{\nu_{2,t}} dW_t^4
\]

\[
d\nu_{1,t} = \left( \eta_1 - \kappa_1 \nu_{1,t} - \kappa_2 \nu_{2,t} \right) dt + \sigma_{\nu1} \sqrt{\nu_{1,t}} dW_t^5
\]

\[
d\nu_{2,t} = \left( \eta_2 - \kappa_{12} \nu_{1,t} - \kappa_{22} \nu_{2,t} \right) dt + \sigma_{\nu2} \sqrt{\nu_{2,t}} dW_t^6
\]

- Incorporate spot and forward cost-of-carry
- Two volatility factors are necessary to fit option prices across the maturity and moneyness dimensions
- Both volatility factors are predominantly unspanned by the futures contracts
- Model does not really capture the time-variation in the skew – short-term and long-term skew variations are separated
Oil Relative Risk Reversals

<table>
<thead>
<tr>
<th>CL3 25RR / V</th>
<th>CL12 25RR / V</th>
<th>CL60 25RR / V</th>
</tr>
</thead>
</table>

![Graph showing oil relative risk reversals from June 21 to February 26, 2009.](image)
Yield curve is endogenous to the model (exogenous for HJM and market models)

Allows to analyse the cross-properties of swap rates and swaptions

Volatility is approximately log-linear in factors

Long-dated swaptions (straddles) are highly correlated to the slope of the yield curve (corr(straddles, 5y-10y) > 0)
Brigo and El-Bachir (2006) credit

\[ \lambda_t = y_t^\beta + \psi(t, \beta) \]

\[ dy_t^\beta = \kappa \left( \mu - y_t^\beta \right) dt + \nu \sqrt{y_t^\beta} dZ_t + dJ_t \]

- Shifted jump-diffusion square-root process for the modelling of default intensity processes fitting single-name default swap term structures and prices of default swaptions
- Allows large CDS implied volatilities and realistic jump features in the default intensity dynamics
- Can be considered for valuation of credit derivatives payoffs and hybrid interest rate / credit payoffs
Literature

- Albanese (2005) *A Stochastic Volatility Model for Risk-Reversals in Foreign Exchange*
- Bates (1996) *Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options*
- Black and Scholes (1973) *The Pricing of Options and Corporate Liabilities*
- Brigo and El-Bachir (2006) *Credit Derivatives Pricing with a Smile-Extended Jump Stochastic Intensity Model*
- Carr and Wu (2007) *Stochastic Skew for Currency Options*
- Christoffersen, Heston and Jacobs (2007) *The Shape and Term Structure of the Index Option Smirk*
- Crosby (2006) *A Multi-factor Jump-Diffusion Model for Commodities*
- Da Fonseca, Grasselli and Tebaldi (2006) *A Multifactor Volatility Heston Model*
- Dupire (1994) *Pricing with a smile*
- Fang (2000) *Option Pricing Implications of a Stochastic Jump Rate*
- Heidari, Hirsa and Madan (2008) *Analytical Pricing of Swaption in Affine Term Structures with Stochastic Volatility*
- Heston (1993) *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*
- Jaeckel (2008) *Stochastic Volatility Models - Past, Present and Future*
- Merton (1976) *Option pricing when underlying stock returns are discontinuous*
- Schwartz and Smith (2000) *Short-Term Variations and Long-Term Dynamics in Commodity Prices*
- Trolle and Schwartz (2008) *Unspanned stochastic volatility and the pricing of commodity derivatives*
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Continuous Probability Distribution

- Cumulative distribution function (CDF)
  \[ F_X (x) = \Pr [X < x] \]

- Probability density function (PDF)
  \[ f_X (x) = \frac{dF_X (x)}{dx} \quad F_X (x) = \Pr [X < x] = \int_{-\infty}^{x} f_X (t) dt \]

- Expected value
  \[ E[g (X)] = \int_{-\infty}^{\infty} g(t) f_X (t) dt \quad E[1] = 1 = \int_{-\infty}^{\infty} f_X (t) dt \]
Let use the Black-Scholes assumptions:

- the log-return from time 0 to time $T$ is normally distributed

$$X = \ln \left( \frac{S_T}{F(0,T)} \right) \sim N \left( \mu, \sigma^2 \right), \quad \mu = -\frac{\sigma^2}{2} \quad \text{and} \quad \sigma^2 = \sigma^2 T$$

$$q_T(x) = \frac{1}{\sigma_T \sqrt{2\pi}} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right) \quad \text{risk-neutral probability density}$$

In general, the risk-neutral expected value of $e^X$, where $X$ is a compensated random variable, is equal to one:

$$E_Q \left[ e^X \right] = E_Q \left[ \frac{S_T}{F(0,T)} \right] = E_Q \left[ S_T \right] \frac{F(0,T)}{F(0,T)} = 1$$
Pricing using RN probability density

- **Long forward position**

\[ f_T(K) = e^{-rT} E_Q \left[ S_T - K \right] = e^{-rT} F_T E_Q \left[ \frac{S_T}{F_T} - \frac{K}{F_T} \right] = \]

\[ = e^{-rT} F_T E_Q \left[ e^X - e^K \right] = e^{-rT} F_T \int_{-\infty}^{\infty} (e^x - e^k) q_T(x) \, dx \]

- **Long call**

\[ c_T(K) = e^{-rT} E_Q \left[ (S_T - K)^+ \right] = e^{-rT} F_T \int_{k}^{\infty} (e^x - e^k) q_T(x) \, dx \]

- **Long put**

\[ p_T(K) = e^{-rT} E_Q \left[ (K - S_T)^+ \right] = e^{-rT} F_T \int_{-\infty}^{k} (e^k - e^x) q_T(x) \, dx \]
Derivation of the Black-Scholes formula

\[
c_T(K) = e^{-rT} E_Q \left[ (S_T - K)^+ \right] = e^{-rT} F_T \int_k^\infty (e^x - e^k) q_T(x) \, dx =
\]

\[
= \int_k^\infty (Se^{yT}e^x - Ke^{-rT}) q_T(x) \, dx = S Q \int_k^\infty e^x q_T(x) \, dx - KP \int_k^\infty q_T(x) \, dx =
\]

\[
= S Q \int_k^\infty e^x \frac{1}{\sigma_T \sqrt{2\pi}} e^{-\frac{(x+\sigma_T^2/2)^2}{2\sigma_T^2}} \, dx - KP \left(1 - N \left(\frac{k + \sigma_T^2/2}{\sigma_T}\right)\right) =
\]

\[
= S Q \int_k^\infty \frac{1}{\sigma_T \sqrt{2\pi}} e^{-\frac{(x-\sigma_T^2/2)^2}{2\sigma_T^2}} \, dx - KP N \left(-\frac{-k - \sigma_T^2/2}{\sigma_T}\right) =
\]

\[
= S Q N \left(\frac{-k + \sigma_T^2/2}{\sigma_T}\right) - KP N \left(\frac{-k - \sigma_T^2/2}{\sigma_T}\right) =
\]

\[
= S Q N \left(\frac{\ln \left(F_T/K\right) + \sigma_T^2/2}{\sigma_T}\right) - KP N \left(\frac{\ln \left(F_T/K\right) - \sigma_T^2/2}{\sigma_T}\right)
\]
Over the Black-Scholes model

- Let say that the asset price (S) follows a Bates-like stochastic volatility process with jumps
  - mean-reverting stochastic volatility process as in the Heston model
  - lognormal jump-in-price component with Poisson arrival time as in the Merton jump-diffusion model

- However, $q_T(x)$ as closed-form risk neutral probability density does not exist anymore!

- New technique is needed.
Characteristic functions

- In probability theory it is the continuous Fourier transformation of the probability density function

\[ \phi_X(u) = E\left[e^{iuX}\right] = \int_{-\infty}^{\infty} e^{iuX} f_X(x) dx \]

- Probability density function is the continuous inverse Fourier transformation of the characteristic function

\[ f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iuX} \phi_X(u) du \]

- For independent random variables

\[ \phi_{X+Y}(u) = E\left[e^{iu(X+Y)}\right] = E\left[e^{iuX} e^{iuY}\right] = E\left[e^{iuX}\right] \cdot E\left[e^{iuY}\right] = \phi_X(u) \cdot \phi_Y(u) \]
Sum of processes

- Pure diffusion process (Black-Scholes)

\[ \varphi_D(u) = \exp\left( -\frac{\sigma^2 T}{2} (iu + u^2) \right) \quad f_D(x) = \frac{1}{\sigma \sqrt{2\pi T}} \exp\left( -\frac{(x - \sigma^2 T/2)^2}{2\sigma^2 T} \right) \]

- Pure jump process (lognormal size, Poisson arrival)

\[ \varphi_J(u) = \exp\left( -\lambda T i (\exp(\eta + \delta^2/2) - 1) u + \lambda T \left( \exp\left((i\eta - u \delta^2/2) u\right) - 1 \right) \right) \]

- Jump-diffusion process (Merton)

\[ \varphi_{D+J}(u) = \varphi_D(u) \cdot \varphi_J(u) \]
Pricing using characteristic functions

- Long call

\[ c_T(K) = e^{-rT} F_T \int_k^\infty (e^{x} - e^{k}) q_T(x) \, dx \]

- Make an adjustment for later purposes

\[ c_T(K) = e^{-rT} F_T e^{-\alpha k} \int_k^\infty (e^{x+\alpha k} - e^{k+\alpha k}) q_T(x) \, dx \]

- Apply the Fourier and then the inverse Fourier transform

\[ c_T(K) = e^{-rT} F_T \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{k}^{\infty} (e^{x+\alpha k} - e^{k+\alpha k}) q_T(x) \, dx \, dk \, dv = \]

\[ = e^{-rT} F_T \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) \, dv = e^{-rT} F_T \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_T(v) \, dv \]
Pricing using characteristic functions

\[
\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} \int_{-\infty}^{\infty} \left( e^{x+\alpha k} - e^{k+\alpha k} \right) q_T(x) \, dx \, dk =
\]

\[
= \int_{-\infty}^{\infty} q_T(x) \int_{-\infty}^{\infty} \left( e^{x+\alpha k} - e^{k+\alpha k} \right) e^{ivk} \, dk \, dx =
\]

\[
= \int_{-\infty}^{\infty} q_T(x) \frac{e^{(\alpha+1+iv)x}}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \, dx =
\]

\[
= \frac{1}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \int_{-\infty}^{\infty} e^{(\alpha+1+iv)x} q_T(x) \, dx =
\]

\[
= \frac{1}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \varphi_T(v - (\alpha + 1)i)
\]

payoff

process
FFT based option pricing

\[ c_T (K) = e^{-rT} F_T \int_0^\infty e^{-\alpha k} \frac{1}{\alpha^2 + \alpha - v^2 + i (2\alpha + 1)v} \phi_T (v - (\alpha + 1)i) dv \]

- \( v = \text{GetV}() \); // Grid in the integration space
- \( k = \text{GetK}() \); // Grid in the log-strike space
- \( \text{data} = \text{GetU}(v) \); // Get the input parameter for the CF
- \( \text{cf} \rightarrow \text{FromUToPhi(data)} \); // Evaluated CF
- \( \text{payoff} \rightarrow \text{FromPhiToPsi}(v, \text{data}) \); // Apply the payoff
- \( \text{ft} \rightarrow \text{FromPsiToIntegrand}(v, \text{data}) \); // Get the integrand
- \( \text{ft} \rightarrow \text{Weightening(data)} \); // Numerical trick for DFT
- \( \text{ft} \rightarrow \text{Transform(data)} \); // Discrete Fourier transformation
- \( \text{payoff} \rightarrow \text{ModifyBack}(k, \text{data}) \); // Reverse the adjustment
- \( \text{ft} \rightarrow \text{Interpolate(data, logStrike)} \); // Interpolate the vector
Direct integration based option pricing

\[
c_T(K) = e^{-rT} F_T \left( 1 - \frac{e^{k/2}}{\pi} \int_0^\infty \frac{dv}{v^2 + 1/4} \text{Re} \left[ e^{-ivk} \phi_T(v - i/2) \right] \right)
\]

- No need anymore for equal grid steps
- Use adaptive quadratures like the adaptive Simpson method
- Adaptive upper bound (I start with upper bound = 62.5)
- Pricing error can be targeted (eg. set to 0.1 vega in calibrations)
- Caching if several strikes are computed at the same time
Control variate for Fourier inversion

\[ c_T (K) = c_T^{BS} (K) + c_T (K) - c_T^{BS} (K) = \]

\[ = c_T^{BS} (K) + e^{-rT} F_T \left( 1 - \frac{e^{k/2}}{\pi} \int_{0}^{\infty} \frac{dv}{v^2 + 1/4} \text{Re} \left[ e^{-ivk} \left( \varphi_T (v - i/2) - \varphi_T^{BS} (v - i/2) \right) \right] \right) \]

\[ \sigma^{BS} = \sqrt{V - M^2} = \sqrt{\left[ -\text{Re} \varphi_T'' (0) \right] - \left[ \text{Im} \varphi_T' (0) \right]^2} \]

- Calculate CF derivatives numerically (eps = 1e-5)
- Better convergence achieved both for FFT and direct integration
Control variate for Fourier inversion

- Heston (var0 = 0.04, varInf = 0.04, kappa = 0.6, lambda = 0.2, rho = -0.5, T = 2)
- StdDev = 29.3%
- NoControl (245 evaluations needed), Control (53 evaluations needed)
Affine jump-diffusion models

\[ dX_t = \left( K_0 + K_1 \cdot X_t \right) dt + \sigma(X_t, t) dW^Q_t + dJ_t \]

\[
\left( \sigma(X_t, t) \sigma(X_t, t)^T \right)_{i j} = H_{0 i j} + H_{1 i j} \cdot X_t 
\]

\[ \Lambda_t = l_0 + l_1 \cdot X_t \]

\[ \theta_\nu(u) = \int \exp(u \cdot z) d\nu(z) \]

\[ H_{0 i j}, l_0 \in \mathbb{R}, K_0, H_{1 i j}, l_1 \in \mathbb{R}^N, K_1 \in \mathbb{R}^{N \times N} \]
### Affine transform

\[ \psi^X (u, X_t, t, T) = E^Q \left[ e^{-\int_t^T (\rho_0 + \rho_1 \cdot X_u) du + u \cdot X_T} \bigg| \mathcal{F}_t \right] = e^{\alpha(t) + \beta(t) \cdot X_t} \]

with alpha and beta satisfying the following complex-valued matrix Riccati equations

\[
\frac{d\beta(t)}{dt} = \rho_1 - K_1^T \beta(t) - \frac{1}{2} \beta(t)^T H_1 \beta(t) - l_1 \left( \theta(\beta(t)) - 1 \right)
\]

\[
\frac{d\alpha(t)}{dt} = \rho_0 - K_0^T \beta(t) - \frac{1}{2} \beta(t)^T H_0 \beta(t) - l_0 \left( \theta(\beta(t)) - 1 \right)
\]

with boundary conditions

\[ \beta(T) = u, \quad \alpha(T) = 0 \]
Affine extended transform

\[
\phi^X (v, u, X_t, t, T) = \mathbb{E}^Q \left[ vX_T \cdot e^{-\int_t^T (\rho_0 + \rho_1 \cdot X_u) du + u \cdot X_T} \left| \mathcal{F}_t \right]\right] = \\
= \psi^X (u, X_t, t, T) \cdot \left( A(t) + B(t) X_t \right)
\]

with \(A\) and \(B\) satisfying the following complex-valued matrix Riccati equations

\[
\frac{dB(t)}{dt} = K^T_1 B(t) + \beta(t)^T H_1 B(t) + l_1 \nabla \theta(\beta(t)) B(t)
\]

\[
\frac{dA(t)}{dt} = K^T_0 B(t) + \beta(t)^T H_0 B(t) + l_0 \nabla \theta(\beta(t)) B(t)
\]

with boundary conditions

\[
B(T) = v, \quad A(T) = 0
\]
Affine characteristic of log-returns

\[
\varphi_T (u) = E^Q \left[ e^{iuX_T} \bigg| \mathcal{F}_t \right] = e^{\alpha (u,t) + \beta (u,t) \cdot X_t}
\]

- Solve the ODEs either analytically or numerically
  - Solving numerically, use control measure to apply adaptive stepsize methods
- The ODEs may become stiff for high value of \( u \)
  - Solving a stiff problem needs more time
  - Avoid using high \( u \) \( \Rightarrow \) use direct integration rather than FFT
  - Use implicit schemes to solve the ODEs
  - Even in case of jumps the derivatives have polynomial form, thus also the Jacobian is polynomial
- Numerical solution for ODEs are competitive with analytical solutions
  - Heston analytical CF: complex square root, complex exp, complex log \( \Rightarrow \)
    these are all expensive functions, while numerical solution involves only basic operations
Explicit method for Trolle-Schwartz ODEs

$u = 800$, initial number of time steps = 20
Implicit method for Trolle-Schwartz ODEs

$u = 800$
Literature

- Carr and Madan (1999) Option Valuation Using the Fast Fourier Transform
- Lewis (2001) A Simple Option Formula for General Jump-Diffusion and Other Exponential Levy Processes
- Kyriakos (2004) Option Pricing Using the Fractional FFT
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Schwartz-Smith calibrated to AHD on 03-Jun-08

Graph showing market volatility and calibrated volatility over time, with a legend indicating different markers for market and calibrated forward volatilities.
Copper (CAD) futures and IVs

![Graph showing Copper (CAD) futures and IVs over time with various labels and markers. The x-axis represents dates from 10-Mar-07 to 10-Mar-09, and the y-axis represents price levels from 2500 to 14500. The graph includes lines for different labels like LPI3, LPI15, LPI27, LPI63, LMCA DS03, LMCA DS15, LMCA DS63, and LMCA DS63. Each line represents different periods and price movements.](image-url)
Primary aluminium (AHD) futures and IVs

![Graph showing primary aluminium (AHD) futures and IVs over time.](image)
Quoting the smile by delta

BF

RR

vol

put delta -25% ATM +25% call delta
FX relative risk reversals
FX relative butterflies

EURUSD 25B3M / V3M
EURUSD 25B1Y / V1Y
EURUSD 25B4Y / V4Y
WTI oil relative risk reversals

-15%  -10%  -5%  0%  5%  10%

07-Jul-08 07-Aug-08 07-Sep-08 07-Oct-08 07-Nov-08 07-Dec-08 07-Jan-09 07-Feb-09

CL3 25RR / V  CL12 25RR / V  CL60 25RR / V
WTI oil relative butterflies

Graph showing the relative butterflies of WTI oil from 07-Jul-08 to 07-Feb-09.
LCO oil 3M and 63M IV smiles in the last half year
19/09/08: strong short smile, skew to right, light long smile, skew to left
20/10/08: high short vol, stronger short smile, no short skew, unchanged long skew
19/11/08: no short smile, clear skew to left, no long smile, curve is rather flat.
19/12/08: further vol increase on short, no change on long
20/01/09: both short term and long-term volatilities are lower, skews are stronger
20/02/09: short smile back, long smile back, very strong long smile on right
20/03/09: no changes
Gold IV term structure dynamics
Silver IV term structure dynamics

![Graph of Silver IV term structure dynamics from 10-Mar-07 to 10-Mar-09](image)
Commodity modelling requirements

- Mean-reversion in asset prices – short-term, long-term
  - Stochastic convenience yield
  - Decreasing volatility term structure

- Multi-factor stochastic volatility – short-term, long-term
  - Volatility smile also on long-term
  - Unspanned stochastic volatility (cannot model the skew)
  - Equilibrium volatility level is stochastic also

- Jumps
  - Discontinuous asset path
  - Closer futures jump larger than longer futures

- Stochastic mean-reverting jump frequency
  - Stochastic implied volatility skew
  - Reduce the need for stochastic volatility
Best model

- There exists no best model, model has to be chosen based on the analysis of the underlying.

- Endogenous and exogenous forward curve models
  - Endogenous: captures the dynamics in a natural way, focuses on risk premia effecting forward curves, allows curve extrapolation.
  - Exogenous: easy to use, fits immediately the current forward curve, but it may be misleading for the dynamics.


- Stochastic leverage for commodities has not been analysed in the literature.

- Pricing using numerically evaluated CFs opens to door to analyse wider range of models.
Literature I

- Andersen, Benzoni and Lund (2004) Stochastic Volatility, Mean Drift, and Jumps in the Short Rate Diffusion
- Borovkova and Geman (2006) Seasonal and stochastic effects in commodity forward curves
- Carr and Wu (2004) Time-changed Lévy processes and option pricing
- Cartea and Figueroa (2005) Pricing in Electricity Markets - a mean reverting jump diffusion model with seasonality
- Casassus and Collin-Dufresne (2005) Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates
- Culot et al. (2006) An Affine Jump Diffusion Model for Electricity
- Deng and Jiang (2005) Levy process-driven mean-reverting electricity price model - the marginal distribution analysis
Literature II

- Ju and Zhong (2006) Fourier transformation and the pricing of average-rate derivatives
- Ribeiro and Hodges (2004) A Two-Factor Model for Commodity Prices and Futures Valuation
- Richter and Sorensen (2002) Stochastic Volatility and Seasonality in Commodity Futures and Options - The Case of Soybeans
- Schwartz and Smith (2000) Short-Term Variations and Long-Term Dynamics in Commodity Prices
1. Modelling Financial Asset Price Dynamics
2. Affine Jump-Diffusion Processes and the Fourier Inversion
3. Modelling the Dynamics of Commodity Curves
4. Pricing Volatility and Jump Derivatives
5. Hedging and Risk Management
6. Questions
Characteristic methods

- Quick and robust, quasi analytical
- In case of direct integration pricing precision can be targeted
- Presence of jumps have no impact on convergence
- Applicable for simple European products like plain vanilla, put, call, digital option, power call, self-quanto
- There are extensions like the FFT convolution to price early-exercise products like Bermudan and American options
- Recent researches to price simple barrier options using FFT
- In practice, the structured products are never simple (e.g., commodity barrier options are usually monitored on the first futures), thus the technique is efficient only for simple European products, but this is what is usually needed for calibrations
Finite difference methods

- Can be quick, but pricing precision cannot be targeted
- Problem with the dimensions (usually up to two, three risk factors can be involved)
- The presence of jumps can slow down the calculations and the convergence a lot (use of PIDE is needed)
- Using local volatility, the number of dimensions may be reduced, thus simple early exercise products not depending on the smile dynamics can be priced efficiently by PDEs
Monte-Carlo methods

- Slower, but efficient tool to price structured products
- Affine jump-diffusion processes can be simulated easily and with good convergence rate
  - In each step we need to simulate only the state variables (<10)
  - Inside a MC step, futures curve can be built from the simulated state variables and the solution of the Riccati equations that we pre-calculated before any MC run
  - Use the Quadratic Exponential scheme to simulate square root processes (stochastic volatility, stochastic jump frequency)
  - If jump frequency is deterministic, simulate the number of jumps up to maturity, then distribute them uniformly according to the Poisson process
- Low-discrepancy random numbers can be used easily for affine processes
- Beside the futures curve, if vanilla prices are needed inside a Monte-Carlo steps, the characteristic methods can be used efficiently (early exercise products)
- Use gap risk methodologies to compensate for jump risks
Agenda presentation

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Hedging and Risk Management

- Calculate sensitivities to the state variables, hedge for a structured product is given by a portfolio of vanilla products that neutralizes the sensitivities.
- In presence of jumps the market is incomplete and there exists no perfect hedge – minimize the residual hedging error.
- Calibrate the affine model globally to historical risk neutral measures → obtain time series for the state variables.
- Historical VaR (1D or 10D) by applying the historical state variable returns on the current state variables.
- Difficult to consider jumps in historical VaR – use stress tests to model the impact of unexpected increase of jump frequencies.
- Project future volatility distributions either from the affine model (risk neutral) or from the historical state variable returns (statistical).
- Counterparty risk management – irrelevant to integrate until we have reliable source for correlation between default and derivatives price.
Literature

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Questions

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