Generic Lévy One-Factor Models for the Joint Modelling of Prepayment and Default: Modelling LCDX

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Abstract

In this paper, we introduce a new robust model for modelling and pricing LCDX tranches. We extend the generic one-factor model of [1], which was developed for modelling and pricing of a synthetic CDO of CDSs, to a model for tranched portfolio of loan-only CDSs (LCDSs). The essential difference is that now also the possibility of prepayments is built in. As a main advantage, the proposed model allows to trade LCDX tranches expressed in base correlations.

Keywords: LCDX tranches, LCDS, loan-only credit default swap, credit risk, credit derivatives, one-factor model, base correlations, tranche pricing, recursive formula, Lévy processes, default risk, prepayment risk, α-stable process, variance gamma process.

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1 Introduction

Credit markets have seen an explosive growth over the last decade. New products, like CDOs, CCPIs, CPDOs, are brought to the market with an unprecedented complexity. In this paper we will deal with structured credit risk products quite similar to the synthetic CDO of credit default swaps (CDSs). More precisely, we will model a tranched portfolio of loan-only credit default swaps (LCDSs). LCDSs are very similar to the more classical products, but with the extra feature of prepayment built in. These LCDSs and portfolios of LCDSs have already found their way into the market. LCDX, for example, is the most popular index of LCDSs and is composed of 100 equally weighted single-name LCDSs. It is the benchmark index for the loan-only CDSs in North America. LCDX was the first standardized liquid product for taking directional views on a portfolio representing the syndicated secured loan market. Moreover, positions can be taken by now in standardized tranches on LCDX: [0% - 5%], [5% - 8%], [8% - 12%], [12% - 15%] and [15% - 100%]. The [0% - 5%] and the [5% - 8%] tranches are quoted as upfront premium (no running spread), while the other tranches are quoted in running spread. Unlike traditional CDSs, LCDS trades are cancellable if no suitable debt remains to deliver upon settlement.[1] LCDS trades are then completely cancelled and LCDX trades are factored down. LCDX tranches will be affected by simply reducing the size of the super-senior tranche. Junior tranches will remain unaffected.

The Gaussian one-factor model is quite frequently used for modelling of CDOs of CDSs. A Monte-Carlo based version of this model extended for LCDX was introduced in [22]. However, the Gaussian model has several drawbacks, one of them is the fact that the model is based on the Normal distribution, a distribution with a very light tail behaviour. Light tails make extreme events very unlikely or even virtually impossible. The very steep base correlation curves represent this fact. In order to match model prices for senior tranches with market spreads one has to blow up correlation very heavily. In [10], a Lévy base correlation alternative was established. Essentially, the Normal light tail behaviour was replaced with a more heavy exponential tail behaviour using the methodology of the generic one-factor Lévy model of [1]. This resulted in much more flatter Lévy base correlation curve with plenty of advantages. However, Monte-Carlo simulations as proposed in [22] may become slow and inaccurate when switching to heavy exponential tails. On the contrary, from a computational point of view, the Lévy base correlation model applied with the recursion formula allows a fast - comparable with the Gaussian one - and accurate pricing and hedging of tranches due to its tractability.

In this paper we extend the generic one-factor model of [1] for the modelling and pricing of a synthetic CDO of CDSs to a model for a tranched portfolio of LCDSs. We propose to use an enhanced recursive formula. The essential difference is that the possibility of prepayments has to be built in as well. The model is generic in the sense that for each mother distribution out of a wide class of distributions (the infinitely divisible ones), one can set up a correlated one factor model in the spirit of firm's value or latent variable thinking. For those Lévy distributions where the correlation as measure does not exist or it is meaningless we define below a more general measure that can be considered as a generalization of the one-factor model. Prepayments are handled very similar to as defaults are handled. Namely, as in the one-factor models for CDOs of CDSs, defaults were triggered if the latent variable representing the firm's value of the reference entity falls below a (time-dependent) low barrier. This low barrier was calculated out of CDS

[1] Here we speak mainly about US LCDSs, which do not terminate upon redemption, repayment or other discharge, but are subject to substitution in such circumstances. This substitution process built into US LCDSs ensures that these contracts are not directly cancelled. However, they can be called in whole in case the substitution is unsuccessful. The European LCDSs were originally issued with callability feature to terminate or cancel the LCDS contract in the case of redemption, loan repayment or other discharge in full. However, since this callability feature caused many difficulties in unwinding trades, the European market decided to issue non-cancellable LCDSs as well. However, even in case of non-cancellable LCDSs there is the risk of prepayment.

quotes in order to match with market implied default probabilities. Here we proceed in a similar way for prepayments. Since the most healthy firms or loans will prepay first, we trigger prepayments by crossing of the same firm's value a high (time-dependent) barrier.[2] Note, that by using a common latent variable for checking whether the default or the prepayment barrier has been reached, a negative correlation between default and prepayment events are intrinsically built in.

Although the model can be theoretically set up for any infinitely divisible distribution, intuitively one should opt for a two-sided distribution, i.e. a distribution with support on the whole real axis. Now, both tails of the distribution play an important role: one tail triggers defaults and the other tail triggers prepayments. Therefore, distributions beside the Normal distribution like the Variance Gamma (VG), CGMY, NIG, Meixner and the α-Stable distributions are now also more in the picture. The Lévy base correlation model in [10] is based on the exponential distribution, a distribution only with support on the positive real axis; and hence seems not really to suit in this situation. Note, that the VG, CGMY, NIG and Meixner distributions all have a similar tail behaviour (on both sides) as the exponential one. More precisely, one can prove that the logarithm of the density of these distributions have a linear decay and this is in contrast with a more fast quadratic decay in case of the Normal distribution. Moreover, the tails of an α-Stable distribution can be even heavier than exponentially decaying.

This paper is organized as follows: we first introduce in Section 2 LCDSs. Next, in Section 3, we set up the generic one-factor model for a tranched portfolio of LCDSs. Then, in Section 4, we present some examples for applicant Lévy processes to use with the proposed one-factor model that we experiment in Section 5.

2 Loan-Only CDS

Loan-Only Credit Default Swaps (LCDSs) are instruments that provide the buyer an insurance against the default of the underlying syndicated secured loan. They are very similar to Credit Default Swaps (CDSs), except that with LCDSs there is a possibility that the loan prepays earlier and hence the instrument is cancellable.

In this article we do not intend to develop a new model for LCDSs, rather we assume that a model for LCDSs is already in use. Regarding the generic one-factor model of [1] for the modelling and pricing of a synthetic CDO it is assumed there that a CDS model is already in use and one is able to extract default and survival probabilities from this model. Similarly, to model a tranched portfolio of LCDSs, here we do not need a full model for LCDSs, but we need a restricted set of information namely survival, prepayment and default probabilities.

During the life of an LCDS contract two kinds of events may be triggered. Either the underlying loan is prepaid, thus the LCDS is cancelled, or the loan-taker goes to default. If neither of these two events has been triggered, we say that the LCDS has survived its term, and it was always in the survival state. If a prepayment event was triggered first, we say that the LCDS has been cancelled. The LCDS was in the survival state until the prepayment date, and it was in the prepayment state after that. If during the life of the LCDS contract a default event was triggered first, we say that the LCDS has defaulted and the LCDS issuer has to pay the recovery adjusted notional amount to the LCDS buyer. In this case the LCDS was in the survival state until the default time and then it was in the default state.

Let us denote with T the maturity of the LCDS, with τ_p the time when a prepayment event is triggered, with τ_d the time when a default event is triggered, with $P_{prepaid}(t)$ the probability

[2] The underlying syndicated secured loan is supposed to be prepaid or refinanced prior to the final maturity when the interest rates are falling or the reference entity has improved its financial position and is therefore able to obtain a better deal on credit spread.

that the loan is prepaid before t and with $P_{defaulted}(t)$ the probability that the loan defaults before t and before any prepayment. Obviously, $0 \leq 1-P_{prepaid}(t)-P_{defaulted}(t) = P_{survived} \leq 1$ is the probability that the loan will be still on track, i.e. has not defaulted nor has been prepaid before t.

$$
\begin{aligned}
P_{prepaid}(t) &= P(\tau_p < t \wedge \tau_p < \tau_d) \\
P_{defaulted}(t) &= P(\tau_d < t \wedge \tau_d < \tau_p) \\
P_{survived}(t) &= P(t < \tau_p \wedge t < \tau_d)
\end{aligned}
$$

It is important to mention that neither $P_{defaulted}(t)$ nor $P_{survived}(t)$ is the same as the corresponding default and survival probabilities of a CDS. Both are conditional on that no prepayment has occurred until t. Also $P_{prepaid}(t)$ is conditional on that no default has occurred until t. However, it may happen that the loan is prepaid before the maturity ($\tau_p < T$), thus the LCDS is cancelled, but the loan-taker goes to default after the prepayment, but before the original LCDS maturity ($\tau_p < \tau_d < T$). In this case, the default event is irrelevant for the LCDS and we say that the LCDS is in the prepayment state between τ_p and T. In the opposite, it is not possible in practice to have a prepayment after a default.

Clearly, at any given time t between 0 and T an LCDS contract is in one of the three possible states (prepaid, defaulted, survived). These three states build up to a continuous time Markov chain, where the state at time 0 is "survived", but the stochastic process which represents the state of the LCDS contract may jump at any time to state "prepaid" or "defaulted". Obviously, it is not possible to step out from the "prepaid" neither from the "defaulted" state. However, the intensity, and so the infinitesimal generator of the continuous time Markov chain may be time-dependent as well as stochastic.

Assuming λ_p to be the intensity that of a prepayment event and λ_d to be the intensity that of a default event, the infinitesimal generator of the continuous time Markov chain at time t may be written as follows.

$$
A(t) = \begin{pmatrix} -\lambda_p(t) - \lambda_d(t) & \lambda_p(t) & \lambda_d(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

The second and the third lines contain zeros, because it is not allowed to step out neither from the prepaid state nor from the defaulted state. In case of an intensity based LCDS model one may model λ_p and λ_d as correlated stochastic processes. An example can be found in [23], where the author proposes affine diffusion models. Nonetheless, also a model with deterministic processes for λ_p and λ_d fits this framework. Starting from the infinitesimal generator of the continuous time Markov chain, it is easy to derive the survival probability process $P_{survived}(t)$, while the prepayment and default probability processes $P_{prepaid}(t)$ and $P_{defaulted}(t)$ are proportional to each other and sum up to $1 - P_{survived}(t)$.

$$
\begin{aligned}
P_{survived}(t) &= E\left[e^{-\int_0^t (\lambda_p(s)+\lambda_d(s))ds}\right] \\
P_{prepaid}(t) &= \int_0^t E\left[\frac{\lambda_p(s)}{\lambda_p(s)+\lambda_d(s)} \cdot \frac{d(1-P_{survived}(s))}{ds}\right] ds \\
P_{defaulted}(t) &= \int_0^t E\left[\frac{\lambda_d(s)}{\lambda_p(s)+\lambda_d(s)} \cdot \frac{d(1-P_{survived}(s))}{ds}\right] ds
\end{aligned}
$$

Now, suppose that the LCDS is with a notional N and the spread for protection is denoted with S. If the protection fees of the LCDS are payable on the dates $t_1, t_2, \ldots, t_n = T$ (if the LCDS is still on track), which dates are typically monthly or quarterly taken, then it results that the present value of the so-called fee leg equals the expected present value of all fees paid.

$$PV_{Fee} = S \cdot N \cdot \sum_{i=1}^{n} E\left[D\left(t_i\right) P_{survived}\left(t_i\right) \Delta t_i + AccC_i\right]$$

The so-called protection leg, the expected present value of the losses in case of default equals (as for a CDS)

$$PV_{Loss} = N \cdot \int_0^T E\left[\left(1 - R\left(t\right)\right) D\left(t\right) \frac{dP_{defaulted}\left(t\right)}{dt}\right] dt,$$

where $R(t)$ is the recovery rate, $D(t)$ denotes the risk-free discount factor from t to time zero and $E[.]$ is the expectation operator. The two definitions above are general enough to allow stochastic recovery and interest rates as well, which can correlate even with the default and prepayment processes. Furthermore, in line with market conventions, if the default happens in between two coupon payment dates, then at the default time accrued coupon ($AccC$) and recovery adjusted notional payment are due.

3 Tranched portfolios of LCDSs

In this section, we make a step further and we extend the well-known one-factor model from [1] to coping with prepayment probabilities.

Let $X = \{X_t, t \in [0,1]\}$ be a Lévy process based on an infinitely divisible distribution: $X_1 \sim L$. Denote the CDF of X_t by $\mathcal{H}_t(x)$, $t \in [0,1]$, and assume it is continuous: $P(X_t \leq x) = \mathcal{H}_t(x)$. Let $X = \{X_t, t \in [0,1]\}$ and $X^{(i)} = \{X_t^{(i)}, t \in [0,1]\}$, $i = 1, 2, \ldots, m$ be independent and identically distributed Lévy processes. All processes are independent from each other and are based on the same mother infinitely divisible distribution L. Furthermore, let $0 < \rho < 1$, be a correlation parameter.

We propose the generic one-factor Lévy model:

$$A_i(T) = X_\rho + X_{1-\rho}^{(i)}, \quad i = 1, \ldots, m.$$

In general, a simple calculation (using the stationary and independent increments property) leads to $\mathrm{Corr}[A_i, A_j] = \rho$. However, in some cases, like for the α-Stable distribution, the second moment of the distribution is not finite. Therefore, a measure like the correlation makes no sense. However, this mathematical issue does not restrict the generic one-factor Lévy model. Just one should consider ρ not as a correlation parameter, but rather as a parameter which measures what is the time fraction of the whole term when the market, thus all the issuers moved together, and what is the time fraction when the issuers moved idiosyncratically. The advantage of applying such kind of generalization in the terminology is, that there is no need anymore to standardize the Lévy processes, and basically any kind of stationary process can be used to copulate.

As in case of the CDO models, the ith loan defaults before t if A_i falls below some preset (time-dependent) barrier $K_i(t)$ (extracted from LCDS and/or CDS quotes to match individual default probabilities). Furthermore, we make the assumption that the ith loan prepays before t if A_i is above some preset barrier $H_i(t)$:

$$P_{defaulted}^{(i)}(t) = P(A_i \leq K_i(t)) \text{ and } P_{prepaid}^{(i)}(t) = P(A_i \geq H_i(t))$$

In order to match default probabilities under this model with default probabilities $P^{(i)}_{defaulted}(t)$ and prepayment probabilities $P^{(i)}_{prepaid}(t)$ observed in the market, set

$$K_i(t) := \mathcal{H}^{[-1]}(P^{(i)}_{defaulted}(t)) \text{ and } H_i(t) := \mathcal{H}^{[-1]}(1 - P^{(i)}_{prepaid}(t)).$$

This model is actually based on the generic Lévy model (Lévy copula) and default is modelled exactly as in the well-known CDO one-factor model. Similar to the idea that if the latent variable A_i is below a preset barrier the firm will default, we now also build in that if A_i is above a high barrier $H_i(t)$, the ith loan will be prepaid.

In Figure 1, a realization of five latent variables A_i, $i = 1, ..., 5$, based on the standard Normal distribution with $\rho = 0.30$ is shown; loan III prepays before $T = 5$, loan IV defaults before $T = 5$ and all the others run until maturity $T = 5$. For each loan, default and prepayment barriers are equal in this example.

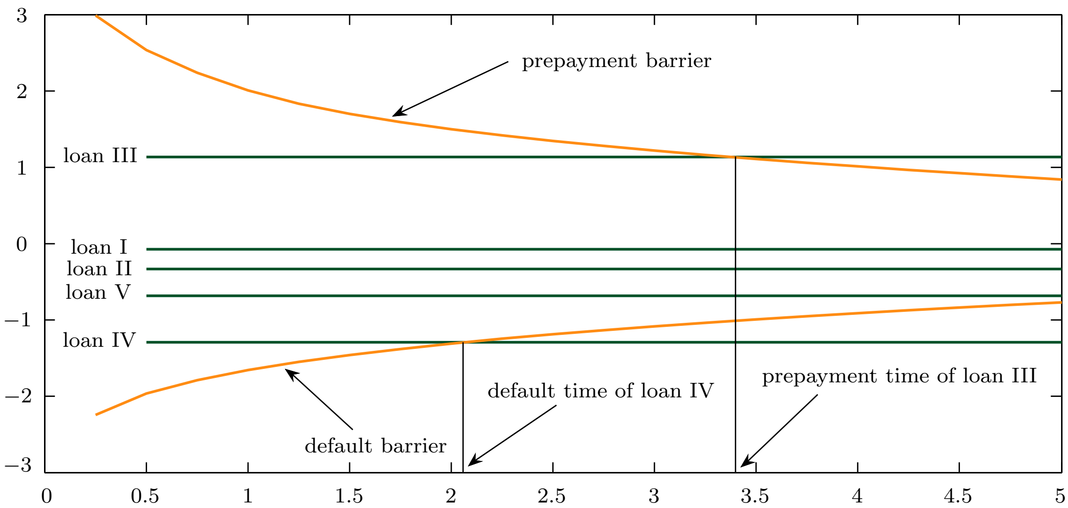

Figure 1: Loan Value Realizations

In order to calculate expected tranche losses and expected outstanding notional of the portfolio of m LCDSs a certain point in time t, we need the joint probability of having k defaults and l prepayments up to time t:

$$\Pi_{k,l}(t) = P(k \text{ defaults and } l \text{ prepayments until } t)$$

Then we have by conditioning on the common factor Y:

$$\begin{aligned}\Pi_{k,l}(t) &= \int_{-\infty}^{+\infty} P(k \text{ defaults and } l \text{ prepayments until } t | Y = y) \mathrm{d}\mathcal{H}_\rho(y), \quad k, l = 0, \dots, m. \\ &= \int_{-\infty}^{+\infty} \Pi^y_{k,l}(t) \mathrm{d}\mathcal{H}_\rho(y), \quad k, l = 0, \dots, m.\end{aligned}$$

Denote by $p_i(y; t)$ the probability that the latent variable A_i is below the default barrier $K_i(t)$ given that the systematic factor X_ρ takes the value y. Similarly, denote by $q_i(y; t)$ the probability that the latent variable A_i is above the prepayment barrier $H_i(t)$ given that the systematic factor X_ρ takes the value y. Then

$$\begin{aligned}p_i(y; t) &= \mathbb{P}(X_\rho + X^{(i)}_{1-\rho} \leq K_i(t)) | X_\rho = y) \\ &= \mathcal{H}_{1-\rho}(K_i(t) - y)\end{aligned}$$

and

$$\begin{aligned} q_i(y;t) &= \mathbb{P}(X_\rho + X^{(i)}_{1-\rho} \geq H_i(t))|X_\rho = y) \\ &= 1 - \mathcal{H}_{1-\rho}(H_i(t) - y). \end{aligned}$$

Denote by $\Pi^{y,n}_{k,l}(t)$ the probability to have k defaults and l prepayments out of n firms conditional on the market factor y at time t. Then by an extension of the classical recursive loss distribution formula, we have

$$\begin{aligned} \Pi^{y,0}_{0,0}(t) &= 1 \\ \Pi^{y,n+1}_{k,l}(t) &= (1 - p_{n+1}(y;t) - q_{n+1}(y;t))\Pi^{y,n}_{k,l}(t) + p_{n+1}(y;t)\Pi^{y,n}_{k-1,l}(t) + q_{n+1}(y;t)\Pi^{y,n}_{k,l-1}(t), \end{aligned}$$

where k and l goes from 0 to n and we assume $\Pi^{y,n}_{-1,l}(t) = \Pi^{y,n}_{k,-1}(t) = 0$ for notational simplicity. Since a loan can not be defaulted and prepaid at the same time, the total number of defaults and prepayments is limited. Basically, in each recursion the formula above for $\Pi^{y,n}_{k,l}(t)$ produces a triangular matrix and necessarily $k + l < n$.

This extension of the classical recursive loss distribution formula leads to the joint unconditional probability to have k defaults and l prepayments out of a group of m firms

$$\Pi_{k,l}(t) = \int_{-\infty}^{+\infty} \Pi^{y,m}_{k,l} \mathrm{d}\mathcal{H}_\rho(y).$$

The marginal distributions for the number of losses and prepayments are given by

$$\Pi^{loss}_k(t) = \sum_{l=0}^{m-k} \Pi_{k,l}(t) \text{ and } \Pi^{prepayment}_l(t) = \sum_{k=0}^{m-l} \Pi_{k,l}(t).$$

These marginal distributions can also be calculated with the univariate recursive loss formulas. For example, denoting by $\Pi^{y,n}_k(t)$ the probability to have k defaults out of n firms conditional on the market factor y at time t. We have

$$\begin{aligned} \Pi^{y,0}_{0,0}(t) &= 1 \\ \Pi^{y,n+1}_{k}(t) &= (1 - p_{n+1}(y;t))\Pi^{y,n}_{k}(t) + p_{n+1}(y;t)\Pi^{y,n}_{k-1}(t), \quad k = 0, ..., n+1. \end{aligned}$$

Then

$$\Pi^{loss}_k(t) = \int_{-\infty}^{+\infty} \Pi^{m,y}_{k} \mathrm{d}\mathcal{H}_\rho(y).$$

Similarly to synthetic CDOs, the expected percentage loss on the portfolio notional at time t is

$$E\left[L(t)\right] = (1 - \bar{R}) \sum_{k=1}^{m} \frac{k}{m} \cdot \Pi^{loss}_k(t);$$

and the expected percentage loss on the CDO tranche $[K_1\% - K_2\%]$ is

$$E\left[L^{Tr}_{K_1,K_2}(t)\right] = \frac{E\left[\min\{L(t), K_2\}\right] - E\left[\min\{L(t), K_1\}\right]}{K_2 - K_1}.$$

The expected percentage decrease on the portfolio notional at time t is

$$E\left[PP(t)\right] = \sum_{l=1}^{m} \frac{l}{m} \cdot \Pi^{prepayment}_l(t) + \bar{R} \sum_{k=1}^{m} \frac{k}{m} \cdot \Pi^{loss}_k(t).$$

The second term in the sum is coming from the fact that defaults amortize the senior-most tranche. Hence, as defaults occur, both the junior-most and the senior-most tranches are impacted. The junior-most tranche notional is reduced by the amount of the loss as discussed above. The senior-most tranche notional is amortized by the amount of the recovery. The expected percentage decrease on the notional of the CDO tranche $[K_1\% - K_2\%]$ is given by

$$E\left[PP^{Tr}_{K_1,K_2}(t)\right] = 1 - \frac{E\left[\min\{1 - PP(t), K_2\}\right] - E\left[\min\{1 - PP(t), K_1\}\right]}{K_2 - K_1}.$$

We can now calculate the fair tranche spreads of a tranche on the portfolio. The fair spread (as for a CDO tranche) is chosen such that the expected present value of the fee payments for that tranche are equal to the expected loss payments.

$$PV_{Fee} = S \cdot N \cdot \sum_{i=1}^{n} E\left[D(t_i)\left(1 - L^{Tr}_{K_1,K_2}(t_i) - PP^{Tr}_{K_1,K_2}(t_i)\right)\Delta t_i + AccC_i\right]$$

and

$$PV_{Loss} = N \cdot \int_0^T E\left[D(t)\,\frac{dL^{Tr}_{K_1,K_2}(t)}{dt}\right] dt.$$

Then S is chosen such that it balances the PV of fee and expected loss legs:

$$S = \frac{\sum_{i=1}^{n} E\left[D(t_i)\left(1 - L^{Tr}_{K_1,K_2}(t_i) - PP^{Tr}_{K_1,K_2}(t_i)\right)\Delta t_i + AccC_i\right]}{\int_0^T E\left[D(t)\,\frac{dL^{Tr}_{K_1,K_2}(t)}{dt}\right] dt}.$$

4 Examples

4.1 Based on the Brownian Motion

The Gaussian one-factor model assumes the following dynamics:

- $A_i(T) = \sqrt{\rho}\,Y + \sqrt{1-\rho}\,\epsilon_i$, $i = 1, \ldots, m$;
- Y and ϵ_i, $i = 1, \ldots, n$ are i.i.d. standard normal distribution with cumulative distribution function Φ.

This model can be casted in the above general Lévy framework. The mother infinitely divisible distribution is here the standard normal distribution and the associated Lévy process is the standard Brownian motion $W = \{W_t, t \in [0,1]\}$.

Indeed, we note that W_ρ follows a Normal$(0,\rho)$ distribution as does $\sqrt{\rho}\,Y$; similarly $W_{1-\rho}$ follows a Normal$(0, 1-\rho)$ distribution as does $\sqrt{1-\rho}\,\epsilon_n$. Adding these independent random variables leads to a standard normally distributed random variable.

4.2 Based on the Variance Gamma Process

The Variance Gamma (VG) distribution with parameters $\sigma > 0$, $\nu > 0$, $\theta \in \mathbb{R}$ and $\mu \in \mathbb{R}$, denoted by $\mathrm{VG}(\sigma,\nu,\theta,\mu)$, is infinitely divisible with characteristic function:

$$\phi_{VG}(u;\sigma,\nu,\theta,\mu) = \exp(\mathrm{i}u\mu)\left(1 - \mathrm{i}u\theta\nu + u^2\sigma^2\nu/2\right)^{-1/\nu}, \quad u \in \mathbb{R}.$$

One can show that this is an infinitely divisible characteristic function. Hence, we can define the VG-process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$, with $\mathbb{P}[X_0^{(VG)} = 0] = 1$ and having stationary and independent VG-distributed increments. To be precise, $X_t^{(VG)}$ has a $\mathrm{VG}(\sigma\sqrt{t}, \nu/t, \theta t, \mu t)$ law. The descriptive statistics of the $\mathrm{VG}(\sigma,\nu,\theta,\mu)$ distribution is given in Table 1.

	$\mathrm{VG}(\sigma,\nu,\theta,\mu)$	$\mathrm{VG}(0, 0.25, -2, 2)$
mean	$\theta + \mu$	0
variance	$\sigma^2 + \nu\theta^2$	1
skewness	$\theta\nu(3\sigma^2 + 2\nu\theta^2)/(\sigma^2 + \nu\theta^2)^{3/2}$	-1
kurtosis	$3(1 + 2\nu - \nu\sigma^4(\sigma^2 + \nu\theta^2)^{-2})$	4.5

Table 1: Mean, variance, skewness and kurtosis of the $\mathrm{VG}(\sigma,\nu,\theta,\mu)$ distribution.

The class of VG distributions has been introduced by Madan and Seneta [14]. A VG-process may also be defined as a Brownian Motion with drift time-changed by a Gamma-process. A number of papers have developed the variance gamma model for asset returns and its implications for option pricing. In equity and interest rate modelling, the VG-process has already proven its capabilities, see e.g. Schoutens [21].

A comparison of the VG distribution with the Normal distribution is made in Figure 2. We chose the VG parameters as given in the second column of Table 1. From the logarithm of the density functions, it is easy to see that the Normal distribution has a quadratic decay on the tails, while the decay of the VG tails is only linear.

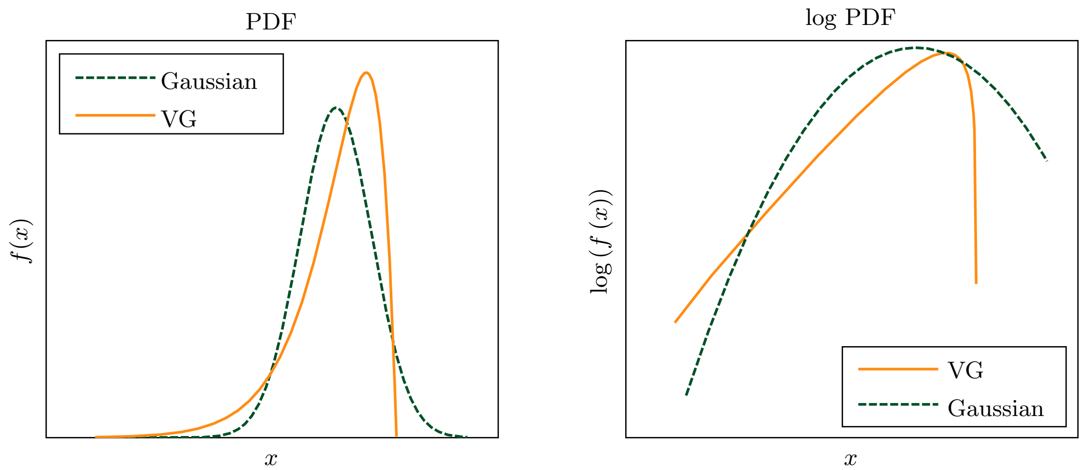

Figure 2: Variance Gamma vs Gaussian Probability Densities

4.3 Based on the Maximally Skewed α-Stable Motion

The class of Lévy Skew α-Stable distributions or just Stable distributions was developed by Paul Lévy. The Stable distributions have the important property of stability: if a number of independent identically distributed (IID) random variables have a Stable distribution, then a linear combination of these variables will have the same Stable distribution, except for possibly different shift and scale parameters. In this manner, Stable distributions owe their importance in both theory and practice to the generalization of the Central Limit Theorem. The Generalized Central Limit Theorem can be applied even to random variables without second (and possibly first) order moments and states that the sum of a number of IID random variables will tend to a Stable distribution. The Normal and the Cauchy distributions are special cases of Stable distributions.

All Stable distributions are infinitely divisible. Except for the Normal distribution, the Stable distributions are heavy-tailed distributions. Here, we will consider the Maximally Skewed α-Stable distribution, which has already proven its capabilities in finance, see e.g. [7]. Restricting the skewness parameter to its maximal, only one parameter remains free to be fixed before calculating the Lévy base correlations. The big advantage is that the heavy-tail and the strong skewness, which are needed in the copulation to be able to fit the market prices, can be modelled with only one free parameter. Further advantage is that for Maximally Skewed α-Stable distributions rapid calculation of the probability density and distribution functions is available. See [18] for details.

The Maximally Skewed α-Stable distribution is characterized by the characteristic function

$$\phi_S(u;\alpha) = \exp(-(\mathrm{i}u)^\alpha \sec(\pi\alpha/2)), \quad 1 < \alpha \leq 2.$$

A comparison with the Normal distribution is made in Figure 3, where the distribution characteristic, α was fixed to 1.5. One may conclude that the α-Stable distribution has even heavier tails than the VG distribution. The extremely fat tail can be characterized by the convexity in the log PDF.

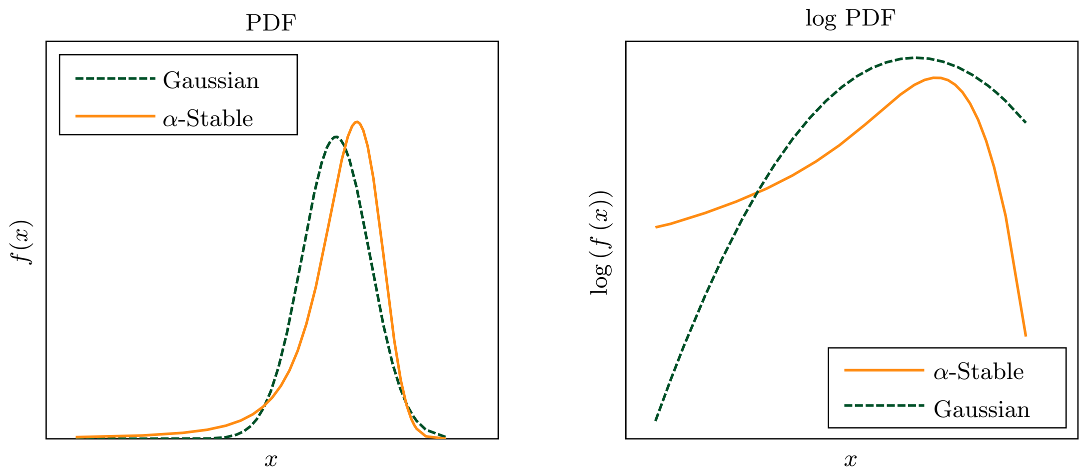

Figure 3: α-Stable vs Gaussian Probability Densities

5 Numerical Experiment

It is widely thought, that when working with LCDSs it is necessary in any way to build up a stochastic intensity model, where the default arrival and prepayment arrival processes are correlated. However, we should keep in mind that for the LCDX model presented in Section 3 we need only a set of restricted data from an LCDS model. Namely, we need only survival, default and prepayment probabilities. If we take a look at the formulation of these probabilities in Section 2, one may conclude, that in order to calculate them, only some kind of integrated processes and their proportion is needed. A deterministic model for the default and prepayment intensities may be considered as an integration of the true stochastic processes. Nevertheless, a deterministic model may be easily fitted to the market by bootstrapping as it is usually done for CDSs.

Therefore, as long as one matches the market quotes and guesses right the proportion of the integrated default arrival and prepayment arrival processes, there is no need for a stochastic LCDS model. If somebody already has a stochastic LCDS model, no problem, it can be attached to the LCDX model presented in Section 3, but if such a model is not available yet, it is still possible to find a robust technique to alternate. This is also how we act in this section. Since the scope of this article does not incorporate a sophisticated LCDS model, for numerical experiments we rather choose a robust deterministic LCDS model in order to represent the behaviours of the proposed LCDX model.

Markit, as data provider, publishes every day CDS, LCDS and LCDX tranche quotes. One may consider to do a simultaneous calibration for the CDS and LCDS markets, like calibrating the default intensity processes based on CDS quotes and calibrating the prepayment processes based on LCDS quotes. However, analyzing the market data one may conclude that the liquidity for the LCDS market is still very low. The CDS and LCDS markets are rather segmented now. Sometimes even the closing LCDS curves are not self-consistent. Moreover, one should pay attention that the difference between the two markets is not only in the prepayment feature. Special care needs to be taken, since recovery assumptions for CDSs (typically around 40%) are different from the typical recovery assumptions for LCDSs (around 70%). Moreover, most of the LCDSs do not default in case of restructuring. Obviously, some part of the CDS spread is the premium related to restructuring, while the bigger part of the spread compensates the expected loss in case of a true default. An interesting study can be found in [5], where the authors compare quotes of CDS contracts with and without restructuring event. They find that the average premium for restructuring risk represents appr. 6 to 8 percents of the swap rate without restructuring.

In this section, we opt for using a deterministic LCDS model similar to the one presented in [16]. We consider that $\lambda_p(t)$ is constant both in time and across issuers, while for each issuer $\lambda_d(t)$ is piecewise-constant and bootstrapped from the corresponding LCDS curve. Based on [8], 38% of loan-takers that issued leveraged loans exited the senior secured loan market within five years of issuing that loan. Since this rate is the historical realization of the prepayment probability and prepayments are usually positively correlated with the market state, we apply — let us say — 5% risk premium and we fix λ_p to ensure that the average probability of prepayment within five years is equal to 39.9%.

Now, having a simplistic LCDS model, we compute LCDX base correlations for the Gaussian, for the Variance Gamma and for the Maximally Skewed α-Stable models. With the Variance Gamma model, we intend to analyse a skewed copula. Therefore, we set up a Variance Gamma process, where the increments have variance equal to 1, skewness equal to -1 and kurtosis equal to 4.5, which is 1.5 times higher than the Gaussian kurtosis. Correspondingly, the Variance Gamma parameters are: $\sigma = 0$, $\nu = 0.25$ and $\theta = -2$. The Maximally Skewed α-Stable model can not be described by variance, skewness and kurtosis, since the higher order moments do not exist. Therefore, intuitively we choose $alpha = 1.5$, which represents a distribution for the increments being in between the Gaussian ($\alpha = 2$) and the Cauchy ($\alpha = 1$) distributions. This parameterization of the Maximally Skewed α-Stable model produces skewed copulation and fat tails.[3]

In Figure 4 and Figure 5 one finds the results for the 5 year LCDX.NA.10 index on 30-Apr-2008 and 30-May-2008. LCDX.NA.10 is tranched in the following way: $[0 - 5\% - 8\% - 12\% - 15\% - 100\%]$. The $[0\% - 5\%]$ and the $[5\% - 8\%]$ tranches are quoted as upfront premium (with no running spread), while all the others are quoted in running spread. Even though the parameters for the Variance Gamma and the Maximally Skewed α-Stable models were not optimized, one may conclude that these Lévy models produce much flatter base correlation curves.

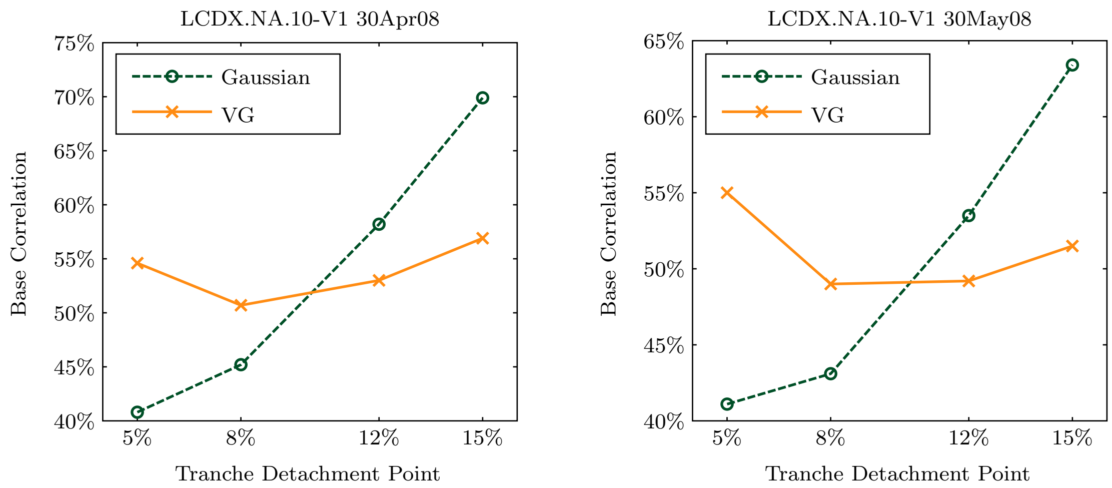

Figure 4: Variance Gamma vs Gaussian Base Correlations

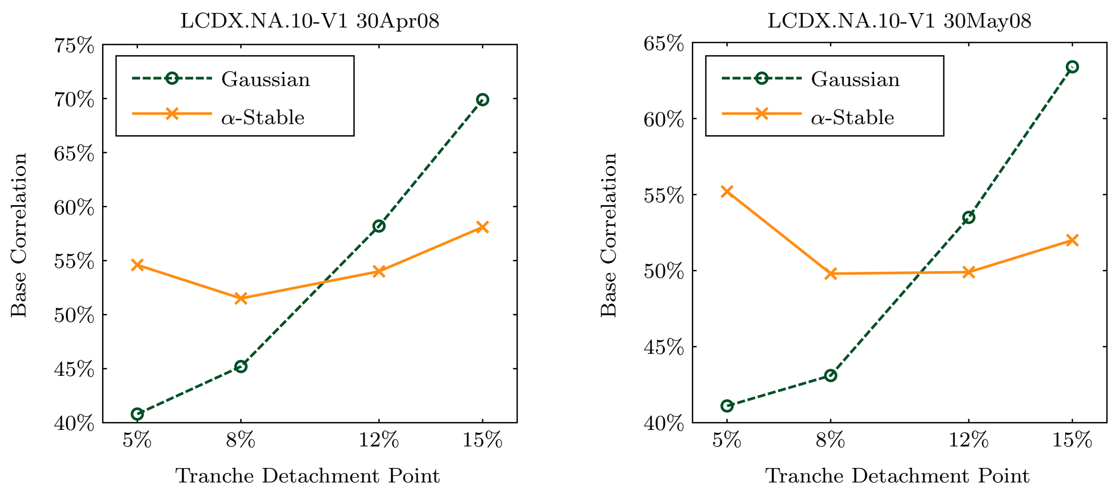

Figure 5: α-Stable vs Gaussian Base Correlations

[3] It is important to mention that this parameterization of the Variance Gamma and the Maximally Skewed α-Stable models is very much in line with the equity markets' implied volatility surfaces. Applying these parameters on the logarithm of stock returns, it produces the well known equity implied volatility smiles, long-term skews and general term structures.

About numerical details, we implemented the whole model in C++. When integrating by the common market factor we use 200 points from the midpoint quadrature as described in [9]. We evaluate the cumulative and the inverse cumulative distribution functions of the Lévy distributions by FFT. To calibrate the base correlation curves we use a Brent's method like optimizer extended with the Müller's method. On a portable computer with a 2Ghz processor the full calibration of a base correlation curve takes appr. 40-60 seconds, where most of the time is spent with the calculation of the enhanced recursive formula, and a fraction of time is needed for the FFT evaluations too.

6 Summary

In this paper, we extended the generic one-factor CDO model with prepayments. We presented a generic framework for LCDS models, and we showed how these models can be linked to our recursive LCDX model. We introduced different Lévy processes to copulate in the LCDX model, and in our numerical experiment we showed that both the Variance Gamma and the Maximally Skewed α-Stable models with some default parameters produce flatter base correlation curves than the generic Gaussian model. As the main result, we delivered a model, which allows to trade LCDX tranches expressed in base correlations. This feature is very favourable in view how the CDO market works as of today.

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