Understanding Volatility Dynamics and its Modelling

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5th November, New York
1. Introduction

2. Conventions and Basic Market Instruments

3. Building Blocks for Volatility Modelling

4. Pricing Techniques for Volatility Derivatives

5. Calibration and Capturing the Volatility Dynamics

6. Empirics and Modelling in Practice

7. Questions and Answers
Before analysing the volatility dynamics, look at the spot price behaviour.

Citibank share price
- Large daily movements were observed in the last 3 years.

What is the one day Value at Risk being long of one share?
Introduction

- Take the daily returns $\ln(\frac{S_t}{S_{t-1}})$ and normalize them by their historical mean and standard deviation.

- Kernel smooth the empirical distribution to display it.

- Highly leptokurtic distribution suggests the presence of jumps.
Introduction

- Take the logarithm of the densities

- For the standard normal distribution:

\[ p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \]

\[ \ln p(x) = -cx^2 \]

- Merton model: simple diffusive asset price with lognormally distributed asset price jumps
Introduction

Daily returns are not identically distributed!

Market activity (volatility) is stochastic either

Normalize the daily returns by the short-term ATM volatility, then standardize them

It results in almost normal error term distribution

Where are the jumps?

C US 01/11/06-30/10/09

Kernel density
Gaussian density

standardized daily log-return / short term vol
Introduction

- 1 day 99% VaR: ~2.5 exc. / year

- Unfiltered VaR
  - Nov07-Oct08: 11 exceptions
  - Nov08-Oct09: 7 exceptions
  - Underestimates the risk before and during the crisis
  - Overestimates the risk getting out of and after the crisis

- Filtered VaR
  - Nov07-Oct08: 3 exceptions
  - Nov08-Oct09: 2 exceptions
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**Conventions and Basic Market Instruments**

- Conventions to quote European options
  - Strike prices (K) and moneyness (y)
  - Sigma quotation (∑)
  - Delta quotation (Δ_{BS})
  - Moments and percentiles

\[
y = \ln \frac{K}{F}
\]

\[
\Sigma = \frac{y}{\sigma_{ATMF} \sqrt{T}}
\]

\[
\Delta_{BS} = -\Phi \left( -\frac{y}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T} \right) \cdot DF
\]
Conventions and Basic Market Instruments

ESTOXX50 30/10/2009

Implied volatility vs. Strike / Forward for different time periods:
- 3M (3 months)
- 1Y (1 year)
- 3Y (3 years)

The graph shows the relationship between implied volatility and the strike or forward price for different time-to-maturity options.
Implied volatility quotation by strike price or by moneyness does not allow to compare the price of skew and smile by maturities or by valuation dates.

Percentiles of implied distribution is expressive, but hardly applicable.

Black-Scholes delta can well describe the statistics of implied distributions.

<table>
<thead>
<tr>
<th>Strike / Fwd</th>
<th>Percentile</th>
<th>Delta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3M</td>
<td>1Y</td>
</tr>
<tr>
<td>50%</td>
<td>4%</td>
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<tr>
<td>150%</td>
<td>100%</td>
<td>91%</td>
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<td>200%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>
Conventions and Basic Market Instruments

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Implied volatility vs. Black-Scholes delta for different time periods:
- 3M
- 1Y
- 3Y

3rd Volatility Trading Summit, IQPC
Conventions and Basic Market Instruments

- In order to capture the dynamics, in course of calibration and modelling focus on key indicators like straddles, risk reversals and butterflies
- By compressing information, we also reduce the noise in our data and we reduce the chance of arriving to local minima in course of calibration
- Take only as much key indicators as much the model can cope with

![Diagram showing volatility and different delta values]
Conventions and Basic Market Instruments

sticky-moneyness

sticky-strike
European vanilla options

\[ \text{payoff} = \max \left( S_T - K, 0 \right) \]

- Payoff depends only on the asset price at maturity
- Derivatives of the terminal density of stochastic price changes
- Bet on the market activity between today and the maturity

Forward starting performance options

\[ \text{payoff} = \max \left( \frac{S_{T_2}}{S_{T_1}} - K, 0 \right) \]

- Payoff depends on the asset price at forward start and at maturity
- Derivatives of the forward density of stochastic price changes
- Bet on the market activity between forward start and the maturity
- Independent from the activity between today and the forward start
Cliquet spreads

\[
payoff = \sum_{i} \max \left( f_i, \min \left( c_i, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right) \right)
\]

- Set of forward starting performance spread options
- Derivatives of forward densities of stochastic price changes
- Forward start $\rightarrow$ mainly delta neutral
- Spread option $\rightarrow$ use strikes to set them vega neutral
- Bet on the implied volatility skew evolution in the future
Conventions and Basic Market Instruments

- **Variance swaps**
  
  \[ \text{payoff} = U \cdot \left( \frac{1}{T} \sum_{i=1}^{n} \ln \frac{S_{t_i}}{S_{t_{i-1}}} - K \right) \]

  - Derivatives of forward densities of stochastic price changes
  - Bet on the market activity in the future without a spot exposure
  - Without caps and floors on the payoff due to the logarithm in the payoff there may be side effects
  - In case of default risk it is also a credit derivative

- **Variance options**
  
  - Non-linear payoff of realized variance
  - Bet on the mean-reversion of market activity
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Building Blocks for Volatility Modelling

- Modelling the volatility dynamics
  - Find risk factors that drives the daily price movements
  - A model should explain the price of basic market instruments (see previous section) not only on one day, but on each day

- A model is a set of
  - State variables (risk factors) that stochastically change by time,
  - Equations that describe the dynamics of state variable changes and
  - Model parameters that parameterize the equations.

- If the model parameters must be changed every day to match the price of basic market instruments, then the dynamics of asset prices and volatilities are not captured well
Building Blocks for Volatility Modelling

- **Diffusion**
  - Black-Scholes (1973): No jumps, market activity is constant
    \[
    \frac{dS_t}{S_t} = \mu dt + \sigma dW_t
    \]
  - Independent and identically distributed daily asset price returns
  - The only state variable is the asset price \( S_t \)
  - Volatility (sigma) is a model parameter, not a state variable
  - Obviously the BS implied volatility surface is flat and constant in time
  - Pure diffusion model does not fit to market prices neither to their dynamics
Building Blocks for Volatility Modelling

- **Finite activity asset price jumps**
  - Merton (1976): Lognormal jumps with constant arrival rate
    \[
    \frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \left( e^J - 1 \right) dN_t
    \]
  - Still independent and identically distributed daily asset price returns
  - Still the only state variable is the asset price \( S_t \)
  - However, the daily asset price return is not normal anymore
  - BS implied volatility skew or smile on short term
  - Independent finite activity jumps, thus no long-term skew or smile
  - Alternative jump distributions are also possible (exp, double exp)
  - Jump-to-default is a special case
Building Blocks for Volatility Modelling

- Infinite activity asset price jumps
  - Lévy processes: Diffusion + Finite activity jumps + Infinite activity jumps
    \[
    \frac{dS_t}{S_t} = \mu dt + \sigma dL_t
    \]
  - Still independent and identically distributed daily asset price returns
  - Still the only state variable is the asset price \( S_t \)
  - Generalization of diffusion processes with jumps
  - BS implied volatility skew or smile on short term
  - For some configurations the variance and higher moments are infinite
    - CLT does not hold anymore, skew and smile also on long-term
    - Be careful: the variance swap price is infinite!
  - Pure Lévy models may fit to market prices on one day, but they cannot capture dynamics, the change in prices and implied volatilities
Building Blocks for Volatility Modelling

- **Stochastic volatility**
  - Heston (1993): Diffusion subordinated to a square root process
    \[
    \begin{align*}
    \frac{dS_t}{S_t} &= \mu dt + \sqrt{\nu_t} dW_t^S \\
    d\nu_t &= \kappa (\theta - \nu_t) + \sigma \sqrt{\nu_t} dW_t^\nu \\
    \langle dW_t^S, dW_t^\nu \rangle &= \rho
    \end{align*}
    \]
  - Daily asset price returns are not identically distributed anymore
  - Two state variables: the asset price \( S_t \) and the variance \( \nu_t \)
  - Not only the spot price, but also the implied volatility surface can have dynamics already
  - Usually there is not only one, but several risk factors driving the shape of the implied volatility surface (stochastic skewness, stochastic variance risk premia, etc.)
Building Blocks for Volatility Modelling

- **Affine processes**
  - Duffie, Pan, Singleton (2000): affine structure of various risk factors

\[
dx_t = \left(K_0 + K_1 \cdot X_t\right) dt + \sigma(X_t) dW^Q_t + dZ_t
\]

\[
Cov\left(X_t\right)_{ij} = H_{0ij} + H_{1ij} \cdot X_t
\]

\[
\Lambda_t = l_0 + l_1 \cdot X_t
\]

where the mean and volatility can be also time-dependent

- Several risk factors can drive the asset price and volatility dynamics
- Also volatility jumps and mean-reversion in asset prices are allowed
- Incorporating squared factors into the vector of state variables, linear-quadratic dependence structure can be defined (Cheng and Scaillet (2007))
Time-changed Lévy processes

- A Lévy process is subordinated to a stochastic clock (activity rate)

\[
dS_t / S_t = \mu dt + \sigma_t dL_t
\]

where the stochastic clock \( \sigma_t \) can be driven by a square root process or by an alternative mean-reverting process correlated with \( dL_t \).

- There may be various diffusion, finite activity and infinite activity jump components subordinated to different business time clocks.
- Although infinite activity jumps may explain a wide range of phenomenon (long-term implied volatility skew without a correlation between asset and variance returns) they may be unrealistic in the financial word (infinite price for variance swaps).
Localization

- Even if we captured the asset price and volatility dynamics by matching the key indicators we targeted, for specific market instruments there may be some price mismatch.
- We may slightly localize model parameters by time and by state variable levels in order perfectly fit the market data on a specific day.

- Dupire (1994): Diffusion process with local volatilities

\[
dS_t / S_t = \mu(S_t, t) dt + \sigma(S_t, t) dW_t
\]

- The local volatility model localize everything without capturing the dynamics at all (it is well known that this model implies unacceptably low forward volatility skewness).
Risk premia

- A model should link not only the prices of various assets on different days (risk neutral measure), but it should link also the risk neutral returns to the historically observed underlying price movements (statistical measure).

- To apply a risk premium in a model is allowed only if some factors are stochastic and correlated or some factors are discontinuous.
  - In the pure Black-Scholes model it is inconsistent to use different volatility in the historical measure and in the risk neutral measure.
  - In a simple stochastic volatility model like Heston the correlation can not bear risk premia, for that purpose we need a stochastic skew model.

- In stochastic volatility models the variance risk premium affects mainly the term structure of implied volatilities.

- In case of jumps higher risk neutral jump intensity may imply higher implied volatility skew.
Building Blocks for Volatility Modelling

- Analyzing time series of volatility surfaces we may identify various risk factors driving the volatility dynamics.

- Defining a set of stochastic differential equations we can build a wide range of models describing volatility dynamics.

- There are two problems at this moment:
  - How can we price the basic market instruments efficiently to be able to calibrate the model?
  - If we have as many stochastic components as key indicators we target, then how can we achieve convergence in calibrating the model parameters?
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Pricing Techniques for Volatility Derivatives

- Derivatives pricing in general
  - Take market scenarios
  - Calculate the derivatives payoffs conditional on the scenarios
  - Calculate the risk neutral probability for each scenario
  - Weight the derivatives cash-flows by the likelihoods

- One possible way is to use Monte-Carlo simulations
  - We may discretize the stochastic equations, create equal probable Monte-Carlo scenarios and price the derivatives, but it is not efficient
  - Mean-reversion and correlation between risk factors implies that we need too small time steps and too many simulations

- For calibrations we need something fast, quasi closed-form
Pricing Techniques for Volatility Derivatives

- In closed-form solutions we try to solve something like:

\[ \pi = \int CF(x) f(x) \, dx \]

where to get the price \( \pi \) we take the conditional derivatives cash-flows \( CF(x) \) and we integrate them by the probability density \( f(x) \)

- For advanced models we do not know the probability density function \( f(x) \), but most of the time we know a transform of it

- We use the transformed probability density for pricing by applying the transform and its inverse in our formula

\[ \pi = F^{-1} \int CF(y) Ff(y) \, dy \]
The characteristic function is the continuous Fourier transform of the probability density function

\[ \varphi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} f_X(x) \, dx \]

- Probability density function is the continuous inverse Fourier transformation of the characteristic function

There exists various numerical techniques to calculate the inverse transform of the integral

- So called characteristic or Fourier methods: eg. FFT, FRFT, COS, …
- Once we have a characteristic function, we can price our basic market instruments efficiently
Pricing Techniques for Volatility Derivatives

- Let say we consider an affine process, where \( \ln\left(\frac{S_T}{S_t}\right) = CX_T \)

\[
\varphi\left(u, C, X_t, t, T\right) = E^Q\left[e^{iuCX_T} \mid \mathcal{F}_t\right] = e^{\alpha(T-t, iuC)+\beta(T-t, iuC)X_t}
\]

with alpha and beta satisfying the following complex-valued matrix Riccati equations

\[
\frac{d\beta(t)}{dt} = -K^T_1 \beta(t) - \frac{1}{2} \beta(t)^T H_1 \beta(t) - l_1 \left(\theta(\beta(t)) - 1\right)
\]
\[
\frac{d\alpha(t)}{dt} = -K^T_0 \beta(t) - \frac{1}{2} \beta(t)^T H_0 \beta(t) - l_0 \left(\theta(\beta(t)) - 1\right)
\]

with boundary conditions

\[
\beta(T) = iuC, \quad \alpha(T) = 0
\]
Pricing Techniques for Volatility Derivatives

- To price forward starting options and variance swaps we look for the forward density, the density of \( \ln \left( \frac{S_{T_2}}{S_{T_1}} \right) \)

\[
E^Q \left[ e^{iu \ln(\frac{S_{T_2}}{S_{T_1}})} \left| \mathcal{F}_t \right. \right] = E^Q \left[ e^{iu \ln(\frac{S_{T_2}}{S_{t}})} e^{-iu \ln(\frac{S_{T_1}}{S_{t}})} \left| \mathcal{F}_t \right. \right] =
\]

\[
= E^Q \left[ e^{iu \alpha} \left| \mathcal{F}_t \right. \right] = E^Q \left[ e^{iu \alpha} \left| \mathcal{F}_{T_1} \right. \right] e^{-iu \alpha} \left| \mathcal{F}_t \right. \right] =
\]

\[
= E^Q \left[ e^{iu \alpha} \left| \mathcal{F}_t \right. \right] =
\]

\[
= e^{\alpha(T_2-T_1,iuc)+\beta(T_2-T_1,iuc)\cdot X_{T_1}}
\]

- In order to price variance swaps, from the characteristic function we calculate second moment of forward returns
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Because of jumps and stochastic time changes the market is incomplete, perfect hedge is not possible

However, assuming arbitrage freeness there exists an equivalent martingale measure, but we have to choose it

We select the equivalent martingale measure (EMM) by model selection and by the way how we calibrate this model

In order to capture the volatility dynamics, it is not enough to look at one single day, we have to look at the history

Assumption: the risk-neutral measure is fixed not only through strikes and maturities, but also through trading days
Calibration and Capturing the Volatility Dynamics

- Enormous number of model parameters and state variables need to be calibrated to capture the dynamics
  - Model parameters (risk prices) are unique for the whole history
  - Only state variables (risk factors) change from day to day
  - For instance taking 4 years of history with yearly 252 days for a model with 4 state variables and 10 model parameters a total of 4042 parameters need to be calibrated

- However, once the model is calibrated, on a daily basis we need to calibrate only the few state variables
Calibration and Capturing the Volatility Dynamics

Step 1
- Choose 5 maturities and 5 reasonable strikes (by BS delta) for each valuation date → 25 vanilla options
- Choose every 2 weeks a total of 80 valuation dates → 3Y
- Total of ~2000 option prices to calibrate to
- Calibrate model parameters and state variables

Step 2
- Involve all valuation dates and recalibrate
- As initial guess use the calibrated model parameters from step 1

Step 3
- Localize: time-dependent drift, risk premia for the current day
Calibration and Capturing the Volatility Dynamics

Stage 1
- Set expectation range for each model parameter (hint for DE)
- Inside the engine normalize and transform the model parameters
  - log, logit, exp, … + apply Feller condition
- Calibration of the model parameters with Differential Evolution
- Inside function evaluations for each valuation date separately calibrate the state variables with Levenberg-Marquardt
- Reset the state variables after each generation/candidate

Stage 2
- Fine tune model parameters with Levenberg-Marquardt
- Calibrate the state variables only when the error function is evaluated, then cache them and use them to evaluate the Jacobian
Risk management

- We captured the volatility dynamics by calibrating the model to historically observed derivatives prices.
- We are in possession of state variable time series, showing changes according to the statistical measure.
- We can use the historically observed returns on the state variables to simulate future scenarios.
- We may simulate long-term evolution by finding a parametric form for the historical state variable returns or by bootstrapping them.

- Calculate hedge ratios in the statistical measure.
- Calculate Value-at-Risk at various horizons.
- Calculate Potential Future Exposure and Credit Value Adjustments.
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Empirics and Modelling in Practice

- One day calibration of the implied volatility surface
- Historical calibration of the implied volatility surface
- Stochastic volatility mean-reversion
- Asset price and volatility correlation – skew on long-term
- Finite activity jumps – skew on short-term
- Infinite activity jumps – skew on long-term
- Stochastic skewness on short-term
- Stochastic skewness on long-term
- Stochastic variance risk premia
- Asset price mean-reversion
- Humped volatility term structure
Empirics and Modelling in Practice

- Value of a derivative is the value of its replication
  - If I fail to describe the future asset price dynamics, I fail to price

- Exotic pricing is extrapolation of available information
  - Not enough to match market price, also match intuition

- Am I hedged? How my daily PnL fluctuate?
  - Did I forecast well the exotic price dynamics?

- Conclusion: Focus needs to be on dynamics and intuition
Questions

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